A new class of quasi-exactly solvable potentials with position dependent mass

Ramazan Koç* and Eser Körcük†
Department of Physics, Faculty of Engineering University of Gaziantep, 27310 Gaziantep, Turkey

Mehmet Koca‡
Department of Physics, College of Science, Sultan Qaboos University,
PO Box 36, Al-Khod 123, Muscat, Sultanate of Oman

(Dated: October 15, 2004)

A new class of quasi exactly solvable potentials with a variable mass in the Schrödinger equation is presented. We have derived a general expression for the potentials also including Natanzon confluent potentials. The general solution of the Schrödinger equation is determined and the eigenstates are expressed in terms of the orthogonal polynomials.

PACS numbers:

In recent years, physical systems with a position dependent mass[1, 2, 3] and quasi exactly solvable(QES) potentials[4] have been the focus of interest. The effective mass models have been used to describe electronic properties of semiconductors, liquid crystals and various other physical systems[5]. In this letter we suggest a method to obtain a general solution of the Schrödinger equation with a position dependent mass.

We start with a general Hermitian effective mass Hamiltonian which is proposed by von Roos[6]

$$H = \frac{1}{4}\left(m^{\alpha}(x)\mathbf{p}m^{\beta}(x)\mathbf{p}m^{\gamma}(x) + m^{\gamma}(x)\mathbf{p}m^{\beta}(x)\mathbf{p}m^{\alpha}(x)\right) + V(x) \tag{1}$$

with the constraint over the parameters: $\alpha+\beta+\gamma=-1$. Depending on the choice of parameters the Hamiltonian(1) can be expressed in different forms[2]. However, we shall keep the general form of the Hamiltonian. Using the differential operator equivalence of momentum operator $\mathbf{p}=-i\frac{d}{dx}$, it is easy to show that the Hamiltonian (1) can be written as

$$\begin{aligned} &-\frac{1}{2m(x)}\frac{d^2\psi(x)}{dx^2} + \frac{m'(x)}{2m^2(x)}\frac{d\psi(x)}{dx} + (V(x)-E)\psi(x) + \\ &\left[(1+\beta)m(x)m''(x) - 2(\beta+1+\alpha(\alpha+\beta+1)m'^2(x)\right]\frac{\psi(x)}{4m^3(x)} = 0 \end{aligned} \tag{2}$$

where E is the eigenvalue of the Hamiltonian (1). Our task is now to obtain a general expression for the potential $V(x)$ such that the Schrödinger equation can be solved quasi-exactly. Without loss of generality, consider the following QES second order differential equation[7],

$$z\frac{d^2\Re(z)}{dz^2} + \left(\ell+\frac{3}{2}+z(b-qz)\right)\frac{d\Re(z)}{dz} + (-\varepsilon+2jqz)\Re(z) = 0 \tag{3}$$

where ℓ, b, q and ε are constants and j takes integer and half integer values. The function $\Re(z)$ is a polynomial of degree $2j$. The differential equation can be obtained by introducing the following linear and bilinear combinations of the generators of the $sl(2,R)$ Lie algebra,

$$\left[J_-J_0 + (\ell+j+1/2)J_- + qJ_+ + bJ_0 + (-\varepsilon+jb)\right]\Re(z) = 0 \tag{4}$$

which is quasi exactly solvable(QES)[4]. The differential realizations of the generators of the algebra is given by[4],

$$J_- = \frac{d}{dz}, \quad J_0 = z\frac{d}{dz} - j, \quad J_+ = -z^2\frac{d}{dz} + 2jz. \tag{5}$$

The function $\Re(z)$ forms a basis for $sl(2,R)$ Lie algebra. The solution of the differential equation(3) which was determined in the paper[7] is in the following form

$$\Re_j(z^2) = \sum_{m=0}^{2j}\frac{(2j)!(2\ell+1)!(\ell+m)!}{2m!(2j-m)!(2\ell+1+2m)!}P_m(\varepsilon)(-qz^2)^m. \tag{6}$$

Here the polynomial $P_m(\varepsilon)$ satisfies the recurrence relation

$$(2j-m)qP_{m+1}(\varepsilon)-(\varepsilon-bm)P_m(\varepsilon)+m(\ell+m+1/2)P_{m-1}(\varepsilon)=0 \tag{7}$$

with the initial condition $P_0(\varepsilon)=1$. The polynomial $P_m(\varepsilon)$ vanishes for $m\geqslant 2j+1$ and the roots of $P_{2j+1}(\varepsilon)=0$ correspond to the $\varepsilon-$eigenvalues of the algebraic Hamiltonian(4). It is well known that the differential equation (3) can be transformed into the form of the Schrödinger equation and several quantum mechanical potentials can be generated. In order to discuss all the potentials related to the differential equation(3), in a unified manner we introduce a variable $z=r(x)$ then the equation (3) takes the form:

$$\frac{r}{r'^2}\frac{d^2\Re(x)}{dx^2}+\frac{1}{r'}\left[\ell+3/2+r(b-qr)-\frac{rr''}{r'^2}\right]\frac{d\Re(x)}{dx}+(-\varepsilon+2jqr)\Re(x)=0 \tag{8}$$

Now lets turn our attention to the effective mass Schrödinger equation(2). In this case both the Schrödinger equation and the QES differential equation (8) include first order differential terms. One can easily transform the effective mass Schrödinger equation into the form of (8). It is convenient to express the eigenfunction $\psi(x)$ in the usual form

$$\psi(x)=-\frac{2r}{r'^2}m(x)e^{-\int W(x)dx}\Re(x). \tag{9}$$

Substituting (9) into (2) and then comparing with (8) we obtain the following expression for the weight function $W(x)$

$$W(x)=\frac{1}{4}\left(\frac{2m'(x)}{m(x)}-\frac{6r''}{r'}+\frac{(1-2\ell-2br+2qr^2)r'}{r}\right) \tag{10}$$

and an implicit expression for the potential function, as follows

$$\begin{aligned} m(x)\left[V(x)-E\right]= \\ \frac{(\beta+1/4+\alpha(\alpha+\beta+1))m'^2(x)}{2m^2(x)}-\beta\frac{m''(x)}{4m(x)}+\frac{3}{8}\left(\frac{r''}{r'}\right)^2-\frac{r'''}{4r'} \\ \left(b^2-(2\ell+8j+5)q+\frac{4\varepsilon+b(2\ell+3)}{r}+\frac{\ell(\ell+1)-3/4}{r^2}-2bqr+q^2r^2\right)\frac{r'^2}{8}, \end{aligned} \tag{11}$$

where r^i is i^{th} derivative of r with respect to x.

At this point we first discuss the special form of the above potential. When we choose $q=0$ the potential is exactly solvable. Under the conditions, $q=0$ and $m(x)=$constant the potential leads to the Natanzon confluent potentials[8]. To obtain the quantum mechanical potentials the function $r(x)$ should satisfy the relation

$$\sqrt{\lambda_0+\lambda_1/r(x)+\lambda_2/r^2(x)}\frac{dr}{dx}=-\sqrt{m(x)}. \tag{12}$$

As for the special cases, $\lambda_0=\lambda_2=0$, the potential corresponds to the radial sextic oscillator potential; $\lambda_1=\lambda_2=0$ to the QES Coulomb potential and $\lambda_0=\lambda_1=0$, to the Morse potential.

For the corresponding special cases we obtain the following mass dependent potentials with some parameters. Let $\lambda_0=\lambda_2=0$ and $\lambda_1=1/4$ then $r(x)=-u^2=-\left[\int\sqrt{m(x)}dx\right]^2$ and the potential takes the form,

$$\begin{aligned} V(x) &= \frac{\ell(\ell+1)}{2u^2}+\frac{1}{2}\left(b^2-(2\ell+8j+5)q\right)u^2+bqu^4+\frac{1}{2}q^2u^6+ \\ &\quad \frac{(\alpha(\alpha+\beta+1)+\beta+9/16)\,m'^2(x)}{2m^3(x)}-\frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{13}$$

This is a family of radial sextic oscillator potential. We have checked that the for choice of $q=0$ and $m(x)=\left(\frac{a+x^2}{1+x^2}\right)^2$ the potential takes the same form as the potential given in the paper[2] and for $m(x)=cx^2$ the potential corresponds to the potential given by Dutra[3]. The eigenvalue of the Schrödinger equation with the potential given in (7) is given by

$$E=\left(\ell+\frac{3}{2}\right)b+2\epsilon. \tag{14}$$

The energy parameter ε is obtained from the recurrence relation(7).

For the cases $\lambda_1 = \lambda_2 = 0$ and $\lambda_0 = 1/4$ the function $r(x) = -2u$ and the potential takes the form

$$\begin{aligned} V(x) \;=\; & \frac{\ell(\ell+1)-3/4}{8u^2} - \frac{4\varepsilon + (2\ell+3)b}{4u} + 2bqu + 2q^2u^2 + \\ & \frac{(\alpha(\alpha+\beta+1)+\beta+9/16)\, m'^2(x)}{2m^3(x)} - \frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{15}$$

This potential represents a family of QES Coulomb potentials. In order to obtain the standard form of the potential one should redefine the parameters. The eigenvalues of the potential is given by

$$E = -\frac{1}{2}((2\ell + 8j + 5)q - b^2) \tag{16}$$

For the last example we choose $\lambda_0 = \lambda_1 = 0$ and $\lambda_2 = 1$ to obtain a family of QES Morse potentials. Then $r(x) = e^{-u}$ and the potential takes the form

$$\begin{aligned} V(x) =& \frac{1}{2}\left(\varepsilon + (\ell/2 + 3/4)b\right) e^{-u} + \frac{1}{2}\left(b^2/4 - (\ell/2 + j + 5/4)q\right) e^{-2u} - \\ & \frac{bq}{4} e^{-3u} + \frac{q^2}{8} e^{-4u} + \\ & \frac{(\alpha(\alpha+\beta+1)+\beta+9/16)\, m'^2(x)}{2m^3(x)} - \frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{17}$$

The standard form of the Morse potential can be obtained by reordering the parameters. The corresponding eigenvalue is given by

$$E = -\frac{1}{8}\left(\ell(\ell+1) + 1/4\right). \tag{18}$$

We have constructed a class of QES potential for the generalized effective mass Hamiltonian without any restriction in the parameters α and β. We have shown that one can obtain a family of potentials, related to the sextic oscillator, QES Coulomb and QES Morse potentials. The method discussed here can be used to obtain other class of potentials which can be related to the hypergeometric Natanzon class potentials.

* Electronic address: koc@gantep.edu.tr
† Electronic address: korcuk@gantep.edu.tr
‡ Electronic address: kocam@squ.edu.om

[1] Milanovic V and Ikanovic Z 1999 J. Phys. A: Math. Gen., **32** 7001; Dekar L, Chetouani L and Hammann T F 1998 *J. Math. Phys.* **39** 5 2551; Levy-Leblond J M 1995 *Phys. Rev. A* **52** 1845; Levy-Leblond J M 1992 *Eur. J. Phys.* **13** 215; Foulkes W M C and Schluter M 1990 *Phys. Rev. B* **42** 11 505
[2] Roy B and Roy P 2002 J. Phys. A: Math. Gen., **35** 3961
[3] de Souza Dutra A and Almelia C A S 2000 *Phys. Lett.* **A 275** 25
[4] Turbiner A V and Ushveridze A G 1987 *Phys. Lett.* A126 181-3; Turbiner A V 1988 *Commun. Math. Phy.* 118 467-74; Gonzalez-Lopez A, Kamran N, and Olver P J 1993 *Commun. Math. Phys.* 153 117-46; Bender C M and Dunne G V 1996 *J. Math. Phys.* 37 6-11;
[5] Serra L I and Lipparini E 1997 *Europhys. Lett.* **40** 667; Barranco M, Pi M, Gatica S M, Hermandez E S and Navarro J 1997 *Phys. Rev. B* **56** 8997; Einevoll G T 1990 *Phys. Rev B* **42** 3497; Morrow R A 1987 *Phys. Rev. B* **35** 8074
[6] Von Roos O 1983 *Phys. Rev. B* **27** 7547
[7] Koc R, Koca M, Tutunculer H 2002 J. Phys. A submitted/A/136189
[8] Chefrour M T, Chetouani L and Guechi L 2000 *Europhys. Lett.* **51**(5) 479